Application of Unilateral Laplace-Finite Mellin Transform to PDE

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ABSTRACT

Integral transform covers a wide range of application in the various areas of Engineering and Science. Excellent to rewarding efforts of researchers there are rapid developments on theory of fractional transform and its application, of that Laplace transform is often applied. Laplace–Mellin Transform may be a comparatively new algorithmic program, it's a good potential within the future in several space of mathematical sciences. In the present paper we described the ideas of Laplace-Mellin transform. We illustrate the application of unilateral Laplace-finite Mellin transform (UL-FM) to partial differential equations .We obtained some new results.

Keyword: - Double unilateral Laplace transform, Mellin transform, unilateral Laplace-finite Mellin transform, partial differential equation, wave equation, heat equation.

1. INTRODUCTION:

There are various cases that no appropriate transform exists. Mellin transform is one type of non-linear transformation that is widely used for its scale invariance property and closely connected to two-sided Laplace transform. Many works about the idea and applications of LM transform have been studied in [4],[7,8]. L Herve et al in [2], had presented a time-domain analysis based on the MLT (Mellin-Laplace transform) Dr. M sing and M saha in [1] have extended Laplace-Mellin Integral Transformation (LMIT) to a class of 'Generalized Function'. S Khairnar et al [5], have been derived the relation between the finite Mellin integral transform with the Laplace transform by using the double Laplace and Fourier – finite Mellin integral. V Gurari [6] has studied parametric integral representations in the theory of Laplace–Mellin transforms and derives integral representations for the remainders of the Laplace–Mellin transforms in terms of the basic kernel. M Saha [3], have been introduced a new cryptographic scheme using Laplace-Mellin transform and the key is the number of multiples of mod 'n'.
Here we extent the application of Laplace-Mellin transform. In section 2 we show the relation between Laplace and Mellin transform .In section 3 we discuss the properties of newly form transform. Some examples showing the applicability of the transform are given in section 4 and lastly we conclude in section 5.

1.1 Preliminaries:

Let f (t) be a function defined on the positive real axis $0 < t < \infty$. The Mellin transform 'M' is the operation mapping the function f into function F defined on the complex plan by the relation

$$M\left[f(t),s\right]=\overline{f}(s)=\int_0^{\infty} t^{s-1} f(t)\,dt \tag{1}$$

Where $K(s,\ t)=t^{s-1}\ ,t\geq 0$

,P;And the double unilateral Laplace transform is given by

$$L^2\left[f(x,y),s,0,\infty,r,0,\infty\right]=\int_0^{\infty}\int_0^{\infty} e^{-(sx+ry)} f(x,y)\,dx\,dy \tag{2}$$

Here this double integral is exists for parameters r > 0 and s > 0.

2. RELATION BETWEEN LAPLACE AND MELLIN TRANSFORMS

We used the change of variable method so that the combined transform converted into the form of Mellin type transform.

On substituting $y = -log\left(\frac{z}{b}\right)$ implies $z = be^{-y}$, $dy = -\frac{dz}{z}$ in equation (2) and changing limits as,

if $y = 0$ then $z = b$ and if $y = \infty$ then $z = 0$ .Gives,

$$L^2\left[f(x,y),s,0,\infty,r,0,\infty\right] = \int_0^\infty \int_0^b b^{-r} e^{-sx} z^{r-1} f(x,z)\,dx\,dz \tag{3}$$

This equation is the relation between unilateral Laplace transform and finite Mellin transform (UL-FM) with parameters $s > 0$ and $r > 0$ in the range $[0,\infty]$ and $[0,a]$, is denoted as- $L^1M_f\left[f(x,z),s,0,\infty,r,0,b\right]$

In next part we verified the properties for this new unilateral Laplace- finite Mellin transform as like Laplace transform properties.

3. PROPERTIES

P1] Property 1: Linearity Property

The unilateral Laplace - finite Mellin transform is a linear operator.

If α, β be any arbitrary constants and $f(x,z)$, $g(x,z)$ are the functions, then

$$L^1M_f\left[\alpha f(x,z)+\beta g(x,z),s,0,\infty,r,0,b\right]$$
$$= \alpha L^1M_f\left[f(x,z),s,0,\infty,r,0,b\right] + \beta L^1M_f\left[g(x,z),s,0,\infty,r,0,b\right] \tag{4}$$

P2] Property 2: Change of Scale Property

If α, β be any scalar and $f(x,z)$ is any function, then

$$L^1M_f\left[f(\alpha x,\beta z),s,0,\infty,r,0,b\right]$$
$$= \frac{1}{\alpha\beta^r} L^1M_f\left[f(p,q),\frac{s}{\alpha},0,\infty,r,0,b\beta\right] \tag{5}$$

P3] Property 3: Power Property

If β be any scalar and $f(x,z)$ is any function, then

$$L^1M_f\left[f\left(x,z^\beta\right),s,0,\infty,r,0,b\right] = \frac{1}{\beta} L^1M_f\left[f(x,t),s,0,\infty,\frac{r}{\beta},0,b^\beta\right] \tag{6}$$

P4] Property 4: First Shifting Property

$$L^1M_f\left[e^{\alpha x} f(x,z),s,0,\infty,r,0,b\right] = L^1M_f\left[f(x,z),s-\alpha,0,\infty,r,0,b\right] \tag{7}$$

P5] Property 5: Derivative Property

1] First order derivative:

$$L^1M_f\left[f_x(x,z),s,0,\infty,r,0,b\right]$$
$$= (s)L^1M_f\left[f(x,z),s,0,\infty,r,0,b\right] - k \tag{8}$$

Where, $k = \int_0^b b^{-r} z^{r-1} f(0,z)\,dz$

2] Second order derivative:

$$\therefore L^1M_f\left[f_{xx}(x,z),s,0,\infty,r,0,b\right]$$
$$=(s^2)L^1M_f\left[f(x,z),s,0,\infty,r,0,b\right]-sk \tag{9}$$

Similarly, $L^1M_f\left[f_{xxx}(x,z),s,0,\infty,r,0,b\right]$
$$=(s^3)L^1M_f\left[f(x,z),s,0,\infty,r,0,b\right]-s^2k \tag{10}$$

In general,

$$L^1M_f\left[f_x{}^n(x,z),s,0,\infty,r,0,b\right]$$
$$=(s^n)L^1M_f\left[f(x,z),s,0,\infty,r,0,b\right]-s^{n-1}k \tag{11}$$

4. APPLICATIONS:

We apply the derivative property for the illustration of **second order partial differential equation** using unilateral Laplace-finite Mellin transforms.

4.1 Example:1 Solve second order PDE

$$\frac{\partial^2 v}{\partial x^2}=T^2\frac{\partial^2 v}{\partial t^2},\ x>0,t>0 \tag{12}$$

With boundary conditions,

$$v(x,0)=T_0$$
$$v(x,t)\to 0 \text{ as } t\to\infty \tag{13}$$

Solution: Taking UL-FM transform to the equation (12) we get,

$$L^1M_f\left[v_{xx}(x,t),s,0,\infty,r,0,b\right]$$
$$=T^2\left\{D_t{}^2L^1M_f\left[v(x,t),s,0,\infty,r,0,b\right]\right\}$$

By using equation (9) we get,

$$(s^2)L^1M_f\left[v(x,t),s,0,\infty,r,0,b\right]-sk$$
$$=T^2\left\{D_t{}^2L^1M_f\left[v(x,t),s,0,\infty,r,0,b\right]\right\} \tag{14}$$

$$\left(D_t{}^2-\frac{s^2}{T^2}\right)L^1M_f\left[v(x,t),s,0,\infty,r,0,b\right]=-\frac{sk}{T^2} \tag{15}$$

This is a linear differential equation with constant coefficient. Its solution is -

The auxiliary equation is $\left(D_t^2 - \frac{s^2}{T^2}\right) = 0 \quad D = \pm \frac{s}{T}$

The roots are real and different .Therefore the complimentary function is –

$$\text{C.F.} = Ae^{\frac{s}{T}t} + Be^{\frac{-s}{T}t} \quad \text{and} \tag{16}$$

$$\text{P.I.} = \frac{-1}{\left(D_t^2 - \frac{s^2}{T^2}\right)} \frac{sk}{T^2} e^{0t} = \frac{k}{s} \tag{17}$$

The complete solution of equation (15) is given by,

$$L^1 M_f \left[v(x,t), s, 0, \infty, r, 0, b \right] = Ae^{\frac{s}{T}t} + Be^{\frac{-s}{T}t} + \frac{k}{s} \tag{18}$$

Applying boundary conditions (13) we get,

$$L^1 M_f \left[v(x,0), s, 0, \infty, r, 0, b \right] = A + B + \frac{k}{s} = T_0 \tag{19}$$

And $\quad A = 0$,Since $v(x,t) \to 0$ as $t \to \infty$, $B = T_0 - \frac{k}{s}$

Hence equation (18) gives,

$$L^1 M_f \left[v(x,t), s, 0, \infty, r, 0, b \right] = T_0 e^{\frac{-s}{T}t} - \frac{k}{s}\left(e^{\frac{-s}{T}t} - 1 \right) \tag{20}$$

This is the required solution of equation (12).

4.1 Example 2. Solving one dimensional wave equation

The high-frequency line equation or one dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \mathrm{C}^2 \frac{\partial^2 u}{\partial t^2} \tag{21}$$

Satisfying the initial and boundary conditions

i] If $t=0$ then $u(x,0)=0$ for all t (22)
ii] If $t = l$ then $u(x, l)=0$ for all t (23)

Solution: Transforming equation (21) by taking UL-FM we get,

$$L^1 M_f \left[u_{xx}(x,t), s, 0, \infty, r, 0, b \right] = \mathrm{C}^2 \left\{ D_t^2 L^1 M_f \left[u(x,t), s, 0, \infty, r, 0, b \right] \right\} \tag{24}$$

By using equation (9) we get,

$$(s^2)L^1M_f\left[u(x,t),s,0,\infty,r,0,b\right]-sk = \mathrm{C}^2\left\{D_t^{\,2}L^1M_f\left[u(x,t),s,0,\infty,r,0,b\right]\right\} \tag{25}$$

$$\left(D_t^{\,2}-\frac{s^2}{\mathrm{C}^2}\right)L^1M_f\left[u(x,t),s,0,\infty,r,0,b\right]=-\frac{sk}{\mathrm{C}^2} \tag{26}$$

Solving this linear differential equation with constant coefficient, we get the complete solution as -

$$L^1M_f\left[u(x,t),s,0,\infty,r,0,b\right]=Ae^{\frac{s}{C}t}+Be^{-\frac{s}{\mathrm{C}}t}+\frac{k}{s} \tag{27}$$

Applying the boundary conditions (22-23), we get

$$0 = A+B+\frac{k}{s}, \text{ and } \quad 0=Ae^{\frac{s}{\mathrm{C}}l}+Be^{-\frac{s}{\mathrm{C}}l}+\frac{k}{s} \tag{28}$$

Solving these two equations we get,

$$B=\frac{k}{s}\left\{\left(e^{\frac{sl}{\mathrm{C}}}-e^{\frac{-sl}{\mathrm{C}}}\right)^{-1}\left(1-e^{\frac{sl}{\mathrm{C}}}\right)\right\}$$

$$\therefore A=\frac{k}{s}\left\{\left(e^{\frac{sl}{\mathrm{C}}}-e^{\frac{-sl}{\mathrm{C}}}\right)^{-1}\left(e^{\frac{sl}{\mathrm{C}}}-1\right)-1\right\}$$

The equation (27) gives,

$$L^1M_f\left[u(x,t),s,0,\infty,r,0,b\right] = \frac{k}{s}\left\{\left\{\left(e^{\frac{sl}{\mathrm{C}}}-e^{\frac{-sl}{\mathrm{C}}}\right)^{-1}\left(e^{\frac{sl}{\mathrm{C}}}-1\right)-1\right\}e^{\frac{st}{\mathrm{C}}} + \left\{\left(e^{\frac{sl}{\mathrm{C}}}-e^{\frac{-sl}{\mathrm{C}}}\right)^{-1}\left(1-e^{\frac{sl}{\mathrm{C}}}\right)\right\}e^{\frac{-st}{\mathrm{C}}}+1\right\} \tag{29}$$

Where, $k=\int_0^b b^{-r}t^{r-1}u(0,t)\,dt$

$$= \frac{k}{s}\left\{\left\{\left(2\sinh\frac{sl}{C}\right)^{-1}\left(e^{\frac{sl}{C}}-1\right)-1\right\}e^{\frac{st}{C}} + \left\{\left(2\sinh\frac{sl}{C}\right)^{-1}\left(1-e^{\frac{sl}{C}}\right)\right\}e^{\frac{-st}{C}}+1\right\} \quad (30)$$

4.2 Solving one dimensional heat equation

The second order partial differential equation

$$\frac{\partial v}{\partial t} = \frac{1}{R}\frac{\partial^2 v}{\partial x^2} \quad (31)$$

Satisfying the initial and boundary conditions,

i] If $x=0$ then $v(t,0)=0$ for all x and (32)

ii] If $x=l$ then $v(t,l)=0$ for all x (33)

Solution: Taking UL-FM transform of equation (31) we get,

$$L^1M_f\left[v_t(x,t),s,0,\infty,r,0,b\right] = \frac{1}{R}D_x{}^2L^1M_f\left[v(x,t),s,0,\infty,r,0,b\right] \quad (34)$$

By using the first order derivative property (9),we get

$$\left(D_x{}^2 - Rs\right)L^1M_f\left[v_t(x,t),s,0,\infty,r,0,b\right] = -Rk \quad (35)$$

This is a linear differential equation with constant coefficient. Its solution using direct method is as follows:

Auxiliary equation is, $\left(D_x{}^2 - Rs\right)=0$ $\quad \therefore D=\pm\sqrt{Rs}$

The roots are real and different. Therefore, the complimentary function is given by

$$\text{C.F.} = Ae^{\sqrt{Rs}\,x} + Be^{-\sqrt{Rs}\,x}, \quad \text{P.I.} = \frac{k}{s} \quad (36)$$

Hence the complete solution of (35) is given by,

$$L^1M_f\left[v_t(x,t),s,0,\infty,r,0,b\right] = Ae^{\sqrt{Rs}\,x} + Be^{-\sqrt{Rs}\,x} + \frac{k}{s} \quad (37)$$

Applying boundary conditions (32-33) and solving the equations for A and B we get the solution as,

$$L^1M_f\left[v_t(x,t),s,0,\infty,r,0,b\right]$$

$$= \frac{k}{s}\left\{\left\{\left(e^{\sqrt{\mathrm{Rs}}\,l} - e^{-\sqrt{\mathrm{Rs}}\,l}\right)^{-1}\left(e^{\sqrt{\mathrm{Rs}}\,l} - 1\right) - 1\right\} e^{\sqrt{\mathrm{Rs}}\,t} + \left\{\left(e^{\sqrt{\mathrm{Rs}}\,l} - e^{-\sqrt{\mathrm{Rs}}\,l}\right)^{-1}\left(1 - e^{\sqrt{\mathrm{Rs}}\,l}\right)\right\} e^{-\sqrt{\mathrm{Rs}}\,t} + 1\right\} \tag{38}$$

$$= \frac{k}{s}\left\{\left\{\left(2\sinh\sqrt{\mathrm{Rs}}\,l\right)^{-1}\left(e^{\sqrt{\mathrm{Rs}}\,l} - 1\right) - 1\right\} e^{\sqrt{\mathrm{Rs}}\,t} + \left\{\left(2\sinh\sqrt{\mathrm{Rs}}\,l\right)^{-1}\left(1 - e^{\sqrt{\mathrm{Rs}}\,l}\right)\right\} e^{-\sqrt{\mathrm{Rs}}\,t} + 1\right\} \tag{39}$$

5. CONCLUSION

Many applications has recognized by using Laplace transform, Mellin transform. But, Laplace-Mellin transform are also raising model images of transform which works like Laplace operator. In our paper we illustrate the application of Laplace- Mellin transform to partial differential equations which will useful for the practical purpose in the engineering field.

6. REFERENCES

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