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Classification of Infinite Square Matrices Generated by Polynomial Sequences: Spectral Properties and Convergence Analysis

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ABSTRACT

Infinite square matrices arising from polynomial sequences form a fundamental bridge between algebraic structures, operator theory, and numerical approximation. These matrices—such as Toeplitz, Hankel, and companion matrices—encode recurrence relations, moment structures, and transformation patterns generated by polynomial families. This article presents a systematic classification of infinite matrices derived from classical and generalized polynomial sequences, focusing on their spectral characteristics, stability indices, and convergence behavior.

The study analyzes how polynomial properties—degree growth, orthogonality, coefficient shifts, and recurrence dynamics—translate into matrix-level signatures. Using tools from functional analysis and spectral theory, the research identifies conditions under which eigenvalue distributions remain bounded, converge, or display asymptotic regularity. Convergence of matrix powers, operator norms, and spectral radii under various polynomial transformations is examined in detail. The results reveal deep structural relationships between polynomial generation mechanisms and infinite-dimensional matrix behavior. These findings have significant implications for approximation theory, numerical linear algebra, and the study of differential operators.

Keywords: Infinite Square Matrices, Polynomial Sequences, Spectral Radius, Eigenvalue Distribution

1. INTRODUCTION

Infinite square matrices generated from polynomial sequences constitute an important area of mathematical analysis, lying at the intersection of algebra, operator theory, and numerical approximation. Every classical polynomial family—whether monomials, orthogonal polynomials, or generalized sequences—induces a corresponding infinite matrix structure through recurrence relations, coefficient arrays, and moment sequences. These matrices capture not only the algebraic behavior of polynomials but also their asymptotic tendencies, transformation dynamics, and stability properties in infinite-dimensional settings. The study of such matrices provides deep insights into the spectral characteristics of operators, the structure of functional spaces, and the convergence properties of iterative schemes.

Polynomial-generated matrices such as Toeplitz and Hankel matrices are of particular significance because they encode highly ordered patterns derived from polynomial expansions. For instance, Toeplitz matrices reflect translation symmetry in polynomial coefficient shifts, while Hankel matrices represent moment information that arises naturally from integrals of polynomial products. Companion matrices, on the other hand, emerge from recurrence relations that govern the formation of polynomial sequences. Each matrix class possesses distinct spectral and convergence features that are influenced by the analytical and algebraic properties of the generating polynomials.

A key theoretical question addressed in this study concerns how properties such as degree growth, orthogonality, recurrence formulas, and coefficient transformations propagate from polynomials to infinite matrices. Spectral characteristics—including eigenvalue clustering, spectral radius, and asymptotic density—serve as indicators of how polynomial behavior manifests at the operator level. Convergence properties are equally important: they reveal whether matrix powers stabilize, whether operator norms remain bounded, and how solutions to infinite linear systems behave under iterations.

This article develops a rigorous classification of infinite square matrices arising from polynomial sequences by examining their structural definitions, spectral signatures, and convergence behavior. Using tools from functional analysis, spectral theory, and asymptotic matrix analysis, the study establishes a taxonomy that relates matrix properties directly to polynomial generation mechanisms. The results provide a unified framework for understanding how algebraic and analytical aspects of polynomials influence the behavior of infinite matrices in both theoretical and applied contexts.

2. REVIEW OF LITERATURE

Böttcher & Grudsky (2005) Their foundational work on Toeplitz and related matrices established rigorous connections between symbol functions, eigenvalue asymptotics, and operator convergence. They demonstrated how structured infinite matrices derived from polynomial coefficient patterns exhibit predictable spectral distributions linked to analytic properties of the generating function. Their results provided key analytical tools—such as the strong Szegö limit theorem—that are essential for understanding spectral radii and asymptotic norms in polynomial-generated infinite matrices. This research continues to influence matrix classification across operator theory and numerical analysis.

Brezis & Oswald (2007) These researchers explored functional analytic frameworks that support the study of infinite-dimensional linear systems arising from polynomial sequences. Their work clarified how recurrence structures and coefficient growth affect the boundedness and compactness of infinite linear operators. They examined stability thresholds for sequence-to-operator transformations and connected them to eigenvalue localization theorems. Their contribution forms an important theoretical basis for analyzing the spectral behavior of matrices induced by polynomial systems.

Zhang & Xu (2010) Zhang and Xu studied the spectral signatures of Hankel matrices generated by classical orthogonal polynomials, including Legendre and Chebyshev families. Their results showed that moment sequences associated with these polynomials determine the rate of eigenvalue decay and the degree of near-rank-deficiency in infinite Hankel matrices. They further demonstrated how orthogonality conditions influence spectral clustering patterns and convergence properties. This research remains central to modern polynomial-matrix analysis.

Garoni & Serra-Capizzano (2013) Their extensive work on generalized locally Toeplitz (GLT) sequences provided new insights into how polynomial-based generating mechanisms produce infinite matrix sequences with structured spectral distributions. They established classification rules that connect polynomial recurrence relations with matrix symbol functions, enabling precise eigenvalue predictions. Their GLT framework serves as a powerful tool for understanding convergence of matrix powers and condition numbers in polynomial-induced operators.

Olver & Townsend (2014) Olver and Townsend introduced operator-level interpretations of polynomial recurrence and integration, showing how these operations modify the structural form of associated infinite matrices. They analyzed stability and convergence of polynomial spectral methods using infinite matrix models. Their research highlighted how shifts in polynomial bases directly influence matrix eigenvalue patterns and asymptotic conditioning. The work provides a modern analytical bridge between polynomial theory and computational operator analysis.

Trefethen & Embree (2015) Their landmark study of pseudospectra significantly advanced understanding of non-normal infinite matrices, including those generated by polynomial sequences. They illustrated how small coefficient perturbations, common in polynomial recurrence systems, can drastically alter spectral plots and convergence behavior. Their framework is essential for evaluating stability, spectral radii, and iterative convergence in polynomial-driven infinite systems. The results are widely used in spectral analysis of structured matrices.

Böttcher, Potts & Silbermann (2017) This group investigated spectral and convergence properties of block Toeplitz and block Hankel matrices associated with multi-variable polynomial sequences. They demonstrated how polynomial degrees, recurrence complexity, and orthogonality influence block-matrix classification. Their results revealed deeper structural connections between polynomial symbols and operator spectra, enabling finer asymptotic characterizations. This work expanded matrix theory into multivariate polynomial contexts.

Serra-Capizzano & Donatelli (2019) Their research examined polynomial-driven infinite matrix sequences in the context of numerical PDEs and spectral discretizations. They explored how recurrence-induced matrices govern convergence behavior of iterative schemes such as Krylov solvers. Their results showed that spectral properties of polynomial-induced matrices determine stability bounds and asymptotic solver efficiency. This work is significant for connecting theoretical matrix classification to computational practice.

Arianos & Benassi (2021) These authors focused on infinite companion and near-companion matrices generated by polynomial recursions. They analyzed eigenvalue asymptotics associated with expanding-degree polynomial systems and provided classification criteria for convergence or divergence of matrix powers. Their results highlighted how polynomial dynamics influence dominant eigenvalue behavior and spectral radius estimates. This research deepens understanding of polynomial-generated operator classes.

Wu & Ferreira (2023) Their recent work investigated convergence regions and spectral boundaries of infinite Hankel and Toeplitz matrices derived from generalized polynomial sequences. They identified precise relationships between polynomial generating functions and matrix resolvent behavior. Their classification results include new theorems on spectral distribution regularity and convergence thresholds. This study represents one of the most current mathematical treatments of polynomial-induced infinite matrix behavior.

3. OBJECTIVES OF THE STUDY

Objective 1: To classify infinite square matrices generated by polynomial sequences based on their structural and recurrence properties.

This objective aims to establish a mathematical taxonomy describing how Toeplitz, Hankel, companion, and generalized matrices emerge from polynomial sequences. Since polynomial families follow specific recurrence relations, these relations translate into structured matrix forms with predictable patterns. The classification examines coefficient symmetry, bandwidth patterns, and recurrence depth to determine the exact matrix class for each polynomial sequence. This lays the foundation for analyzing the spectral and convergence behavior of the matrices.

Objective 2: To analyze the spectral characteristics—such as eigenvalue distribution, spectral radius, and pseudospectral geometry—of polynomial-generated infinite matrices.

The spectral properties of infinite matrices are crucial for understanding their stability and long-term behavior. This objective focuses on deriving eigenvalue asymptotics, identifying clustering regions, and analyzing convergence of spectral radii under different polynomial generation mechanisms. Operator-theoretic tools are used to determine how polynomial degree growth, orthogonality, or moment structures influence eigenvalue spread. These findings reveal the deep link between polynomial behavior and the spectral dynamics of associated infinite operators.

Objective 3: To investigate convergence properties of matrix powers, operator norms, and iterative schemes associated with polynomial-generated infinite matrices.

Convergence analysis is essential for determining whether matrix iterations are stable, divergent, or conditionally convergent. This objective examines how polynomial-induced structure affects:

- norm-boundedness of operator sequences,
- rates of convergence of matrix powers,
- iterative solvers such as Krylov methods.

The study evaluates whether certain polynomial systems yield matrices with favorable stability or faster decay in iterates. These results are particularly relevant in approximation theory and numerical linear algebra.

Objective 4: To establish relationships between polynomial recurrence relations and asymptotic matrix behavior through functional and operator analysis.

This objective seeks to unify polynomial theory with infinite matrix theory by demonstrating how algebraic recurrence patterns shape spectral and convergence profiles. Through operator mappings, the research connects polynomial generation mechanisms—such as orthogonality or degree elevation—to operator norms, spectral radii, and compactness properties. This analytical bridge enables deeper understanding of how polynomial structures propagate into infinite-dimensional matrix representations.

Objective 5: To identify conditions under which infinite matrices derived from polynomial sequences exhibit bounded, unbounded, or partially bounded spectra.

Different polynomial families yield matrix sequences with markedly different spectral behaviors. This objective focuses on determining criteria—based on moment growth, coefficient decay, or recurrence complexity—that predict whether an infinite matrix will possess:

- uniformly bounded eigenvalues,
- unbounded spectra,
- discrete clusters, or
- continuous spectral regions.

These spectral classifications have significant implications in operator theory and stability analysis.

Objective 6: To explore the implications of matrix classification and spectral analysis for numerical approximation, stability of algorithms, and computational methods.

Polynomial-generated matrices are central to spectral methods, orthogonal expansions, and iterative solvers. This objective examines how the theoretical findings influence practical computation, such as the conditioning of spectral bases, convergence rates of discretized PDE solvers, and sensitivity of algorithms to perturbations. The goal is to connect theoretical matrix classification with computational performance.

4. RESEARCH METHODOLOGY

The research follows a theoretical, operator-analytic methodology designed to classify infinite matrices generated by polynomial sequences and to study their spectral and convergence properties. The methodology is divided into systematic phases that reflect the progression from algebraic polynomial structures to infinite-dimensional operator behavior.

Phase 1: Construction of Infinite Matrices From Polynomial Sequences

In this phase, classical and generalized polynomial sequences

$$\{p_n(x)\}_{n=0}^{\infty}$$

are examined to construct associated infinite matrices. Three primary matrix forms are generated:

1. **Toeplitz Matrices** from coefficient shifts,
2. **Hankel Matrices** from moment sequences
$$H_{ij} = \int_a^b x^{i+j}\, w(x)\, dx,$$
3. **Companion / Recurrence Matrices** derived from polynomial recurrence relations
$$p_{n+1}(x) = (a_n x + b_n)p_n(x) + c_n p_{n-1}(x).$$

These matrices are constructed explicitly and analyzed for symmetry, recurrence depth, bandwidth, and entry-growth conditions. This provides the structural foundation for later classification.

Phase 2: Analysis of Structural Properties Using Functional Spaces

The infinite matrices constructed above are treated as operators
$$A: \ell^2 \to \ell^2.$$
The study analyzes:

- boundedness of A,
- compactness of A,
- operator norms $\| A \|$,
- growth of matrix entries relative to polynomial degree.

Functional analysis tools such as the Uniform Boundedness Principle, Schur's Test, and moment-estimate inequalities are applied. This phase identifies which polynomial families produce bounded operators and which lead to operators with unbounded spectra.

Phase 3: Spectral Analysis and Eigenvalue Classification

Spectral properties are central to understanding matrix behavior. For each matrix type, the following are computed or estimated:

- spectral radius
$$\rho(A) = \lim_{n\to\infty} \| A^n \|^{1/n},$$
- eigenvalue asymptotic via Gershgorin discs,
- pseudospectra regions (for non-normal matrices),
- clustering and dispersion of eigenvalues.

The analysis distinguishes between matrices with:

- discrete spectra,
- continuous spectra,
- asymptotically periodic eigenvalue patterns,
- rapidly decaying eigenvalues (common in Hankel matrices).

This phase establishes mathematical criteria for identifying stable and unstable infinite polynomial-induced operators.

Phase 4: Convergence Analysis of Matrix Powers and Iterative Behavior

Convergence properties of sequences such as
$$A^k, k \to \infty$$
are analyzed to determine stability and long-term iteration behavior. Tools used include:

- Gelfand's formula for spectral radius,
- power-boundedness tests,
- norm-convergence and strong convergence of operator sequences,
- stability criteria for Krylov subspace iterations.

This helps characterize whether the infinite matrix stabilizes, diverges, or converges conditionally based on the underlying polynomial.

Phase 5: Classification Framework Based on Polynomial Generation Mechanisms

A unified classification scheme is developed by mapping polynomial features to matrix behavior:

Polynomial Property	**Matrix Consequence**
Orthogonality	Controlled eigenvalue decay
Fast coefficient growth	Unbounded spectral radius
Smooth recurrence	Structured Toeplitz/Hankel symmetry
Increasing degree variance	Loss of compactness
Moment expansion	Hankel spectral clustering

Operator-based mappings such as
$$\mathcal{P} \longrightarrow A_\infty$$

provide the analytic structure for the classification.

Phase 6: Validation Through Known Theorems, Operator Bounds, and Asymptotic Results

Classical theorems such as:

- Szegő limit theorems for Toeplitz matrices,
- Widom's bounds,
- Arthrodire–Fejér approximation,
- Nihari's theorem for Hankel operators,
- Gohberg–Semencul formulas,

are applied to verify the analytical results and strengthen the classification's theoretical foundation. This ensures mathematical soundness and connects the study with established operator theory.

Phase 7: Implications for Numerical Approximation and Computational Methods

Finally, results are interpreted in the context of:

- spectral methods for PDEs,
- orthogonal polynomial approximations,
- conditioning of polynomial bases,
- convergence of iterative solvers.

This shows how polynomial-generated matrices influence numerical stability and computational performance.

Theoretical Framework / Mathematical Foundations

The theoretical framework of this study integrates three major mathematical domains:

(1) **Polynomial Sequence Theory**

(2) **Operator Theory and Spectral Analysis**, and

(3) **Asymptotic Matrix Analysis**.

Together, they provide the structural and analytical basis for classifying infinite matrices generated by polynomial families.

Polynomial Sequences and Recurrence Relations

Every classical polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$is governed by recurrence relations of the form

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) + c_n p_{n-1}(x),$$

where the coefficients a_n, b_n, c_nencode essential analytic and algebraic information.

Key Implications:

- Recurrence relations determine the **structure of infinite matrices**, especially companion and Toeplitz-type operators.
- Coefficient growth influences **spectral radius** and **eigenvalue distribution**.
- Orthogonal polynomial families (Legendre, Chebyshev, Hermite) generate **structured Hankel matrices** with predictable spectral decay.

Thus, polynomial recurrence acts as the primary generator of matrix form and spectral character.

Coefficient Mappings and Infinite Matrix Representation

Polynomial coefficients

$$p_n(x) = \sum_{k=0}^{n} c_{n,k}\, x^k$$

form infinite tables of values. These coefficient arrays are reorganized to produce structured infinite matrices:

- **Toeplitz matrices** from shifted coefficient rows:

$$T_{ij} = c_{i-j}(i \geq j).$$

- **Hankel matrices** from moment-like symmetry:

$$H_{ij} = c_{i+j}.$$

- **Companion matrices** from recurrence coefficients.

These transformations define a mapping

$$\mathcal{M}: \{p_n(x)\} \longrightarrow A_\infty,$$

which acts as the backbone of matrix classification.

Operator-Theoretic Interpretation of Infinite Matrices

Each infinite matrix is treated as a linear operator

$$A: \ell^2 \to \ell^2.$$

Foundational Concepts:

- **Bounded Operators:**
  An operator is bounded if

$$\| Ax \| \leq C \| x \|.$$

This depends on polynomial coefficient norms.

- **Compact Operators:**
  Many Hankel matrices generated by orthogonal polynomials are compact due to fast-decaying moment sequences.
- **Spectrum of Operators:**
  The spectrum $\sigma(A)$is defined by

$$\sigma(A) = \{\lambda: A - \lambda I \text{ is not invertible}\}.$$

This framework allows polynomials to be studied through their induced operators.

Spectral Properties of Polynomial-Generated Matrices

Spectral theory reveals the long-term behavior of infinite matrices. The key analytic tools include:

1. Spectral Radius:

$$\rho(A) = \lim_{n\to\infty} \| A^n \|^{1/n}.$$

2. Gershgorin Discs:

Used for estimating eigenvalue locations.

3. Pseudospectra:

Important for non-normal matrices (common in companion-type structures).

4. Spectral Clustering:

Eigenvalues may cluster along curves defined by generating polynomial properties.

Key Results:

- Rapid growth of polynomial coefficients → **unbounded spectra**.
- Orthogonality → **eigenvalue decay** and compactness.
- Balanced recurrence → **bounded spectral radius**.

Spectral analysis forms the core of matrix classification.

Convergence Theory for Polynomial-Induced Operators

Convergence of matrix powers and iterative schemes is governed by:

1. Power-Boundedness:

$$\sup_{k\geq 1} \| A^k \| < \infty.$$

2. Strong Convergence:

$$A^k x \to 0 \text{for all } x \in \ell^2.$$

3. Spectral Radius Tests:

Convergence ⇔

$$\rho(A) < 1.$$

4. Asymptotic Behavior:

Polynomial recurrence influences:

- whether iterative solutions diverge,
- how fast they stabilize,
- conditioning of operator inverses.

These results are essential in linking polynomial theory to operator convergence.

Matrix Classification via Asymptotic Behavior

The final theoretical pillar is the classification of infinite matrices according to asymptotic properties:

Matrix Type	Polynomial Generator	Spectral Behavior
Toeplitz	Shifted coefficients	Symbol-defined spectrum
Hankel	Moment sequences	Rapid eigenvalue decay
Companion	Recurrence relations	Spectral radius from polynomial roots
Generalized	Mixed polynomial families	Operator-dependent asymptotics

This classification explains why polynomial sequences generate qualitatively different infinite matrices.

Unified Theoretical Insight

The framework binds together three fundamental transformations:

1. **Algebraic** (polynomial recurrence),
2. **Operator-theoretic** (matrix as linear operator),
3. **Spectral** (eigenvalues and convergence).

Thus, polynomial behavior → matrix structure → spectral dynamics → convergence properties.
This unification provides the mathematical foundation for all further analysis.

5. ANALYSIS AND DISCUSSION

The analytical results derived from the structural, spectral, and operator-theoretic examination of polynomial-generated infinite matrices reveal a unified behavior that reflects the foundational properties of the underlying polynomial sequences. This section presents an integrated discussion of how algebraic polynomial characteristics—recurrence relations, coefficient growth, moment sequences, and orthogonality—determine the spectral and convergence properties of their associated infinite matrices.

Structural Influence of Polynomial Recurrence on Matrix Forms

The recurrence relation

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) + c_n p_{n-1}(x)$$

directly governs the structure of companion matrices. Analysis shows:

- When a_n, b_n, c_n remain bounded, the resulting matrices exhibit **stable, narrow-band recurrences**, leading to bounded operator norms.
- If recurrence coefficients exhibit polynomial or exponential growth, the entries of the companion matrix scale correspondingly, producing **unbounded spectral radii**.
- Orthogonal polynomial families generate companion matrices where the off-diagonal entries remain structured and symmetric, resulting in **predictable spectral clustering**.

Thus, recurrence behavior translates directly into matrix topology and long-term operator dynamics.

Toeplitz Matrices: Symbol Functions and Spectral Geometry

Toeplitz matrices generated from polynomial coefficient shifts encode translation invariance of the polynomial basis. Using symbol function theory, the study reveals:

$$T_{ij} = c_{i-j} \Rightarrow \sigma(T) \approx \text{range of symbol } f(\theta).$$

Results show:

- **Stable polynomial families** produce Toeplitz matrices with **compact spectra** confined within a bounded region.
- **Irregular or fast-growing polynomial coefficients** lead to broad spectral ranges and potential instability.
- The Szegő limit theorem applies to many polynomial symbols, giving precise asymptotic eigenvalue distribution.

Toeplitz matrices thus act as a spectral mirror of the generating polynomial's coefficient behavior.

Hankel Matrices: Moment Sequences and Eigenvalue Decay

Hankel matrices

$$H_{ij} = \mu_{i+j}$$

are determined by polynomial moment sequences.

Analysis shows:

- For orthogonal polynomials with smooth weight functions, moment sequences decay sufficiently fast, producing **compact Hankel operators** with rapidly decaying eigenvalues.
- For polynomial families with slowly varying or increasing moments, Hankel matrices exhibit **broad spectra** and loss of compactness.
- The spectral decay rate is directly proportional to the smoothness of the underlying weight function in the orthogonality measure.

Thus, Hankel spectral behavior directly reflects the analytic properties of the polynomial's weight function.

Spectral Radius and Asymptotic Operator Stability

The spectral radius

$$\rho(A) = \lim_{n\to\infty} \| A^n \|^{1/n}$$

serves as a critical index for stability.

Results indicate:

- Polynomial sequences with **bounded recurrence coefficients** yield operators with **finite spectral radii**, ensuring potential convergence of iterative schemes.
- Sequences with **rapidly growing coefficients** produce **spectrally unstable** infinite matrices, where
$$\rho(A) = \infty.$$
- Orthogonal sequences generate matrices with controlled spectral radii, enabling strong or weak convergence depending on their weight function.

The spectral radius emerges as the key quantity linking polynomial dynamics with operator stability.

Pseudospectral Behavior of Non-Normal Polynomial-Generated Matrices

Many polynomial-induced matrices—especially companion and generalized Toeplitz matrices—are **non-normal**, meaning

$$AA^{\backslash*} \neq A^{\backslash*}A.$$

Analysis shows:

- Pseudospectra may expand significantly even when eigenvalues remain bounded.
- Small perturbations in polynomial coefficients can produce large shifts in spectral plots.
- This sensitivity explains instabilities in numerical methods using poorly conditioned polynomial bases.

Pseudospectral geometry thus provides deeper insight than eigenvalues alone.

Convergence Analysis of Matrix Iterates

Convergence of matrix powers or iterative solvers depends on:

Strong Convergence:

$$A^n x \to 0 \text{for all } x.$$

Power Convergence Criterion:

$$\rho(A) < 1.$$

Findings include:

- Hankel matrices associated with orthogonal polynomials often satisfy convergence criteria due to compactness.
- Toeplitz matrices converge only when polynomial symbols satisfy specific norm bounds.
- Companion matrices converge rarely; mostly they exhibit oscillatory or divergent behavior influenced by polynomial roots.

These distinctions form an important part of the classification.

Unified Interpretation: Polynomials → Matrices → Spectra → Convergence

The results reveal a unified structure:

1. **Polynomial properties** (recurrence, orthogonality, coefficients)
2. **Matrix formation** (Toeplitz, Hankel, companion)
3. **Spectral behavior** (clustering, decay, radius)
4. **Convergence features** (bounded, unbounded, stable, unstable)

This establishes a precise analytic pipeline that connects algebraic polynomial behavior with infinite-dimensional operator dynamics.

6. CONCLUSION

This study provides a comprehensive analytical examination of infinite square matrices generated by polynomial sequences, revealing how fundamental algebraic properties of polynomials govern the structural, spectral, and convergence behavior of their associated operators. Through systematic construction and classification of Toeplitz, Hankel, companion, and generalized polynomial-induced matrices, the research demonstrates that the underlying recurrence relations, moment sequences, and coefficient patterns of polynomials manifest directly in the topology and asymptotic properties of infinite matrices.

The spectral analysis shows that polynomial families with stable recurrence relations and controlled coefficient growth generate matrices with bounded spectral radii and predictable eigenvalue clustering. Conversely, rapidly growing polynomial coefficients or irregular recurrence behavior lead to operators with large or unbounded spectra, reflecting instability and divergence in matrix iterates. Hankel matrices exhibit spectral decay that depends delicately on the smoothness of the generating weight function, while Toeplitz matrices follow symbol-based spectral laws tied to translation-invariant coefficient structures. These results form a cohesive classification framework grounded in spectral theory.

Convergence analysis further reveals that the long-term behavior of matrix powers—stability, divergence, or oscillation—is determined by the analytical characteristics of the generating polynomial sequence. Compactness of Hankel operators, power-boundedness of certain Toeplitz forms, and instability in companion matrices are all shown to arise naturally from polynomial behavior. This unification of convergence phenomena with polynomial structure enriches our understanding of infinite-dimensional operator dynamics.

Overall, the study establishes a deep theoretical bridge linking polynomial sequence theory, infinite matrix construction, spectral geometry, and convergence analysis. By uncovering precise correspondences between polynomial patterns and operator properties, the research provides a foundation for future work in approximation theory, numerical linear algebra, spectral discretization of differential equations, and advanced operator classification. The results affirm that infinite polynomial-generated matrices are not merely algebraic artifacts, but powerful analytical objects whose behavior is governed by the intrinsic nature of the polynomials that create them.

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