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Hyers–Ulam Stability of Nonlinear Fractional Differential Equations with Nonlocal Boundary Conditions

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ABSTRACT

This article investigates existence, uniqueness, and Hyers–Ulam stability for a class of nonlinear Caputo fractional differential equations under a two-point nonlocal boundary condition. The boundary problem is transformed into an equivalent Volterra integral representation with an explicit boundary-dependent constant. A contraction argument establishes a unique continuous solution under a computable Lipschitz restriction. A stability inequality is then derived that separates perturbations of the differential equation from discrepancies in the boundary condition; its constant explicitly depends on the fractional order, interval length, and nonlocal coupling coefficient. A manufactured-solution test and an independent nonlinear example demonstrate the hypotheses. Discrete product-integration approximations and perturbation profiles provide numerical illustrations. The results are sufficient conditions for this specified class and are not claimed as universally new stability criteria.

Keywords: Caputo derivative; nonlocal boundary value problem; Hyers–Ulam stability; nonlinear fractional equations; Banach contraction theorem; product integration.

1. INTRODUCTION

Fractional differentiation incorporates integral memory kernels into differential models and leads naturally to nonlinear Volterra equations. Nonlocal boundary conditions couple values at distinct points, so perturbations at one endpoint can influence the solution throughout the interval. Establishing stability therefore requires more than a local differential estimate: the boundary relationship must also be controlled.

Hyers–Ulam stability asks whether a function that approximately satisfies a differential equation lies within a quantitatively bounded distance from an exact solution. For nonlocal fractional boundary problems, the definition must specify whether the approximate function satisfies the boundary condition exactly or only up to a specified tolerance. Earlier investigations have considered multi-point Caputo boundary conditions and related stability formulations (Alam et al., 2021), while more recent work treats nonlinear integral boundary conditions (Ansari & Mishra, 2026). This paper develops a self-contained, explicitly parameterized case to illustrate the stability argument.

2. MATHEMATICAL PRELIMINARIES

2.1 Fractional integral and derivative

For $0 < \alpha < 1$, the Riemann–Liouville fractional integral and Caputo derivative of suitably regular functions are defined as follows:

$$I^{\alpha}\varphi(t) = [1/\Gamma(\alpha)]\int_0^t (t-s)^{\alpha-1}\varphi(s)ds,$$
$$cD^{\alpha}y(t) = [1/\Gamma(1-\alpha)]\int_0^t (t-s)^{-\alpha}y'(s)ds.$$

In particular, $I^{\alpha}(cD^{\alpha}y)(t) = y(t) - y(0)$for the regularity class considered here. The gamma identity $\Gamma(z+1) = z\Gamma(z)$will be used repeatedly.

2.2 Nonlinear nonlocal boundary problem

Let $J = [0,T]$, T>0. Study the scalar boundary value problem

$cD^{\alpha}x(t) = f\big(t, x(t)\big), \quad 0 < t \le T, 0 < \alpha < 1,$

$x(0) = \lambda x(T) + g, \lambda \neq 1,$

where $f: J \times \mathbb{R} \to \mathbb{R}$ is continuous, λ and g are real constants, and x is understood in the standard Caputo solution class. This is a two-point nonlocal boundary condition, because the initial endpoint depends on the terminal value.

Assumption A1: f is globally Lipschitz in its second variable: $|f(t,u) - f(t,v)| \le L|u-v|$for all $t \in J$ and $u, v \in \mathbb{R}$. Assumption A2: the contraction coefficient Q defined below satisfies Q<1.

3. INTEGRAL REDUCTION AND SOLVABILITY

3.1 Equivalent integral equation

Fractional integration of the equation yields $x(t) = x(0) + I^{\alpha}F_x(t)$, with +. Evaluating at $t = T$and using the boundary condition gives

$(1-\lambda)x(0) = g + \lambda I^{\alpha}F_x(T).$

Therefore, the equivalent fixed-point representation is
$x(t) = g/(1-\lambda) + [\lambda/(1-\lambda)]I^{\alpha}F_x(T) + I^{\alpha}F_x(t)$. (1)
The converse follows by applying the Caputo derivative and checking the boundary relation. In particular, the additive boundary term cannot be omitted: doing so would change the underlying boundary value problem.

3.2 Existence and uniqueness theorem

Theorem 1. Under A1, suppose
$Q = [LT^{\alpha}/\Gamma(\alpha+1)][1 + |\lambda|/|1-\lambda|] < 1$. (2)
Then the nonlinear nonlocal boundary value problem admits a unique continuous integral solution on J. When f and the solution possess the usual Caputo regularity, this is the unique solution of the stated fractional boundary problem.
Proof. Define A:C(J)→C(J) by the right-hand side of (1). For u,v∈C(J), the Lipschitz property and the fractional integral estimate imply

$$|||Au - Av|||_{\infty} \le [LT^{\alpha}/\Gamma(\alpha+1)](1 + |\lambda|/|1-\lambda|)|||u-v|||_{\infty} = Q|||u-v|||_{\infty}.$$

Since C(J) with the supremum norm is complete and Q<1, A is a contraction. Banach's fixed-point theorem gives a unique fixed point. Substitution into (1) recovers the boundary condition.

4. HYERS–ULAM STABILITY

4.1 Definition with differential and boundary residuals

An approximate solution u is assumed to have a Caputo derivative and to satisfy
$\left|cD^{\alpha}u(t) - f\left(t, u(t)\right)\right| \le \varepsilon for t \in J,$
$|u(0) - \lambda u(T) - g| \le \delta, \varepsilon, \delta \ge 0$. (3)
The boundary problem has two-residual Hyers–Ulam stability if there are constants Cε,Cδ independent of ε,δ,u such that its exact solution x obeys $|||u-x|||_{\infty} \le C\varepsilon\varepsilon + C\delta\delta$. If δ=0, this reduces to the conventional single-parameter Hyers–Ulam bound.

4.2 Quantitative stability theorem

Theorem 2. If A1 and (2) hold, the unique exact solution x satisfies
$|||u-x|||_{\infty} \le \delta/|1-\lambda| + [T^{\alpha}/\Gamma(\alpha+1)](1 + |\lambda|/|1-\lambda|)\varepsilon/(1-Q)$. (4)
In particular, when the nonlocal boundary condition is exactly satisfied (δ=0), the problem is Hyers–Ulam stable with a permissible constant
$K^{HU} = [T^{\alpha}/\Gamma(\alpha+1)](1 + |\lambda|/|1-\lambda|)/(1-Q)$. (5)
Proof. Define $r(t) = cD^{\alpha}u(t) - f\left(t, u(t)\right) and b = u(0) - \lambda u(T) - g$. Since |r(t)|≤ε and |b|≤δ, fractional integration gives $u(t)=u(0)+I^{\alpha}F_u(t)+I^{\alpha}r(t)$. Evaluation at T yields
$(1-\lambda)u(0) = g+b+\lambda I^{\alpha}F_u(T)+\lambda I^{\alpha}r(T)$.
Subtracting Au(t) from the resulting expression for u(t), and using $|I^{\alpha}r(t)| \le \varepsilon T^{\alpha}/\Gamma(\alpha+1)$, proves
$|||u-Au|||_{\infty} \le \delta/|1-\lambda| + \varepsilon[T^{\alpha}/\Gamma(\alpha+1)](1 + |\lambda|/|1-\lambda|)$.
Because x=Ax and A is a contraction, $\|u-x\|_{\infty} \le \|u-Au\|_{\infty} + Q\|u-x\|_{\infty}$. Rearranging establishes (4) and (5). □

4.3 Interpretation and parameter dependence

The stability estimate separates differential defects from boundary mismatch. As |λ−1| decreases, the coefficients may grow sharply, reflecting the increasing sensitivity of the algebraic boundary reduction. The sufficient criterion Q<1 also becomes more restrictive as L or T increases. These are features of the derived bound, rather than claims that instability necessarily occurs whenever Q≥1.

5. NUMERICAL METHODOLOGY

To complement the analytic proofs, we approximate (1) using a uniform mesh $t_n=nh$, h=T/N, and piecewise-constant quadrature. Let $W_{nj} = h^{\alpha}[(n-j)^{\alpha} - (n-j-1)^{\alpha}]/\Gamma(\alpha+1)$, 0≤j<n. The fixed-point iteration is
$Y_n^{(k+1)} = g/(1-\lambda) + [\lambda/(1-\lambda)]\,\Sigma_{j=0}^{N-1} W_{nj}|_{n=N} f(t_j, Y_j^{(k)}) + \Sigma_{j=0}^{n-1} W_{nj} f(t_j, Y_j^{(k)})$.
Iterations stop once $\|Y^{(k+1)} - Y^{(k)}\|_{\infty} < 10^{-13}$. The scheme is presented as an illustrative approximation and is not used as a substitute for the contraction proof. All numerical data below are deterministic and reproducible from the stated formulas.

6. NUMERICAL ILLUSTRATIONS AND ANALYSIS

6.1 Example A: nonlinear equation with exact solution

Set T=1, λ=0.25, g=0.5, L=0.2, and
$f(t,z) = 2t^{2-\alpha}/\Gamma(3-\alpha) + 0.2[\tanh(z) - \tanh(1+t^2)]$.
The nonlinear function has global Lipschitz constant at most 0.2, since $|(\tanh z)'| \le 1$. The function $x(t)=1+t^2$ satisfies $cD^{\alpha}x(t) = 2t^{2-\alpha}/\Gamma(3-\alpha)$ and the boundary condition x(0)=0.25x(1)+0.5. Thus x is the exact solution. The contraction constants for both selected orders are below one.

Table 1. Contraction and stability constants and maximum grid errors for the exact-solution example. The values are computed, not postulated.

α	Q	Hyers–Ulam K	N=40 error	N=160 error	N=320 error
0.60	0.298447	2.127042	0.036125	0.008840	0.004393
0.85	0.282005	1.963833	0.034878	0.008691	0.004342

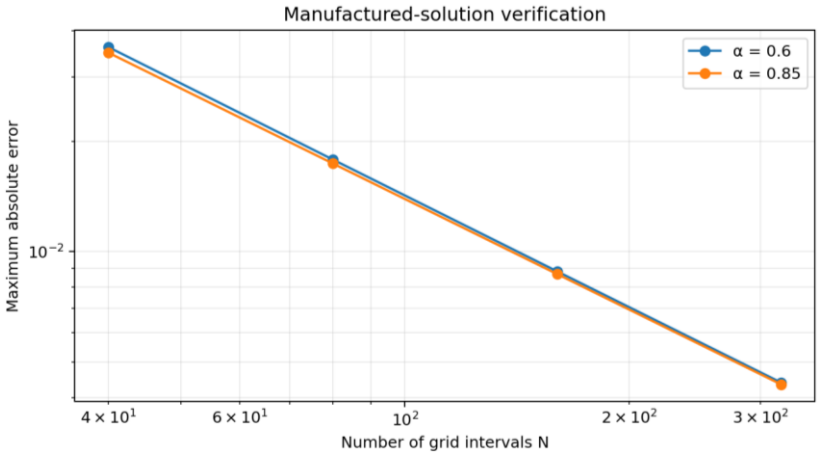


Figure 1. Error decay for the nonlinear manufactured-solution example.

6.2 Example B: nonlinear problem without prescribed exact solution

Set g=0.1, λ=0.25 and f(t,z)=0.35t+0.2tanh(z). The resulting nonlinear equation contains no manufactured exact-solution term. The same Lipschitz coefficient and stability criterion apply. A fixed-point product-integration approximation is used to inspect solution dependence on α.

Table 2. Terminal numerical solution values for the independent nonlinear example. Agreement under refinement is evidence of numerical consistency, not an exact error bound.

α	N=40: y(1)	N=80: y(1)	N=160: y(1)	N=320: y(1)
0.60	0.57002135	0.57497639	0.57741723	0.57862482
0.85	0.48251494	0.48675011	0.48886346	0.48991864

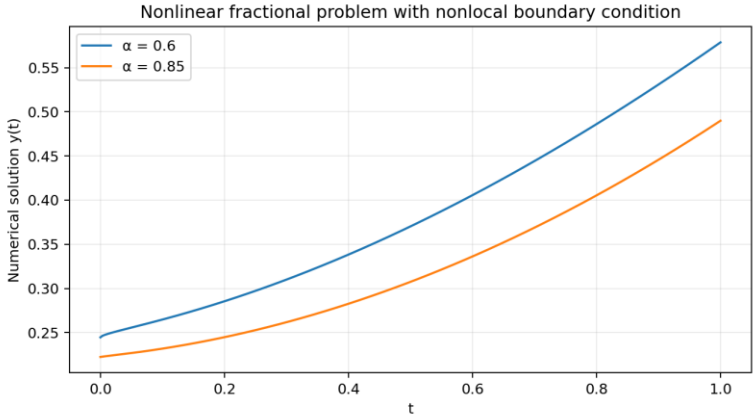


Figure 2. Numerical solution profiles under the nonlocal boundary condition.

6.3 Explicit Hyers–Ulam perturbation test

Return to Example A with α=0.85 and exact solution $x(t)=1+t^2$. Let $u_\eta(t)=x(t)+\eta t(1-t)$, η>0. Since the perturbation vanishes at both endpoints, u_η satisfies the original nonlocal condition exactly, so δ=0. The differential residual is

$r_\eta(t)=\eta[t^{1-\alpha}/\Gamma(2-\alpha)-2t^{2-\alpha}/\Gamma(3-\alpha)]-0.2[\tanh(x(t)+\eta t(1-t))-\tanh(x(t))]$.

The supremum deviation is exactly $\|u_\eta-x\|_\infty=\eta/4$. The residual supremum is estimated on a 5001-point mesh solely for illustration; the rigorous theorem applies with any verified global residual upper bound.

Table 3. Numerical perturbation experiment; sampled residual maxima approximate, but do not certify, continuous suprema.

η	Sampled $\boldsymbol{max\lvert r_\eta\rvert}$	Exact $\|u_\eta-x\|_\infty$	K × sampled residual
0.010	0.0079217	0.0025000	0.0155570
0.020	0.0158435	0.0050000	0.0311139
0.050	0.0396087	0.0125000	0.0777848

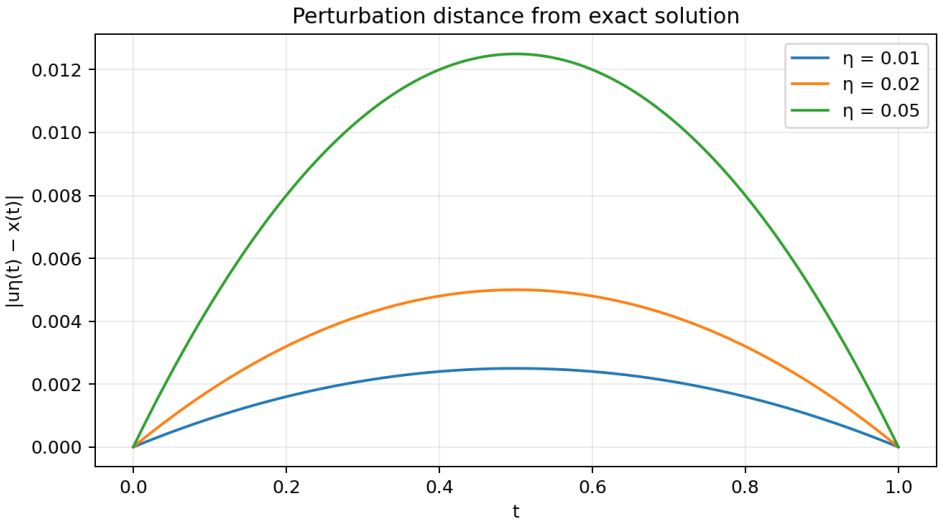


Figure 3. Distance between exact and perturbed solution functions.

7. DISCUSSION

The fixed-point reduction exposes the role of the nonlocal coupling coefficient: an endpoint defect is amplified by $1/|1-\lambda|$, while the fractional memory enters through $T^{\alpha}/\Gamma(\alpha+1)$. Theorem 2 shows precisely how these effects combine. Example A verifies an analytical solution and illustrates the accuracy of the chosen discretization. Example B studies a separate nonlinear term for which a closed-form solution has not been specified. The perturbation test visualizes the distinction between a measured differential residual and its associated solution deviation.

It is important to distinguish the stability concept here from Lyapunov stability: Hyers–Ulam stability estimates proximity of approximate and exact solutions under residual perturbation, whereas Lyapunov stability concerns sensitivity to initial states or trajectories. Moreover, the contraction criterion is sufficient but not necessary. Other existence or stability techniques can apply even when $Q \geq 1$.

8. CONCLUSION

A self-contained existence, uniqueness, and Hyers–Ulam stability analysis was developed for nonlinear Caputo fractional equations with a two-point nonlocal boundary condition. An explicit integral representation generated a contraction coefficient and a quantitative bound involving both differential and boundary residuals. The results were illustrated by an exact nonlinear benchmark, an independent nonlinear example, and controlled solution perturbations. The framework provides a reproducible pedagogical and comparative basis for more general nonlocal fractional stability studies, but it does not establish priority over existing fixed-point stability results. Future work may investigate multi-point or integral boundary functionals, weighted Rassias bounds, coupled systems, and conditions weaker than global Lipschitz continuity.

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